Critical behavior of the majority voter model is independent of transition rates

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We study the critical properties of the majority voter model by using two different transition rates: the Glauber rate and the Metropolis rate. The model with the Glauber rate has been found to be mapped to the majority voter model with noise [de Oliveira, J. Stat. Phys. **66**, 273 (1992)]. The critical temperature and the critical exponents for the two transition rates are obtained from a Monte Carlo simulation with a finite size scaling analysis. The critical temperature is found to depend on the transition rate, but the critical exponents do not. The values of the critical exponents obtained indicate that the model belongs to the same universality class as the Ising model, regardless of the type of transition rate.

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I. INTRODUCTION

The opportunity for choice in purchasing a product or in selecting a representative is rising as the society is becoming more complex. Agents in a market or an election observe their neighborhood to determine their opinion. Recently, opinion dynamics based on a stochastic spin model has been studied actively using physical notions such as nonequilibrium phase transitions [1–3]. The majority voter model is a well-known spin model for opinion dynamics, and has been studied by many researchers [4–15].

Agents on a lattice in the majority voter model gather the opinion of their neighborhood, and change their opinion according to a majority of votes. When the noise parameter q is introduced to put a stochastic process into the majority rule, a minority opinion can be selected with probability q and a phase transition occurs at the critical noise parameter q_c [4–6]. The majority voter model is known to be a nonequilibrium model. Nevertheless, it belongs to the same universality class as the two-dimensional Ising model because of the up-down symmetry [5,16,17].

The choice of transition rate satisfying the detailed balance condition is not unique. Various types of transition rate are applied to study the critical properties of the equilibrium [18,19] and the nonequilibrium [20] models. The critical properties of the equilibrium models do not depend on the transition rate. However, for the nonequilibrium models, the dependence of the critical properties on the type of transition rate is not known well. It is thus fruitful to investigate the critical properties of a nonequilibrium model and their dependence on the transition rate.

In this paper, we have carried out Monte Carlo simulations to study the critical properties of the majority voter model on a square lattice. We use the Glauber and the Metropolis rates using suitably defined "energy" functions (see Sec. II B) in our simulations. We show that the model with Glauber rates is exactly mapped to the original majority voter process with noise. We measure the order parameter, the susceptibility, and Binder's fourth-order cumulant [21] to obtain the critical temperature and the critical exponents, using a finite size scaling analysis. We find that the critical temperature depends on the transition rate, but the critical exponents do not. The values of critical exponents obtained indicate that the majority voter model belongs to the same universality class as the Ising model in two dimensions, independent of the type of transition rate. Finally, by measuring the energy histogram ratio (see Sec. II D), we confirm that the majority voter model does not follow the Boltzmann distribution for either rate.

II. BACKGROUND

A. Majority voter model

The original majority voter model [8] is a spin model in which each site i is occupied by a spin with value either $\sigma_i = +1$ or -1. We consider the model on a square lattice with periodic boundary conditions. The transition rate from a spin configuration $\{\sigma\}$ to another configuration $\{\sigma'\}$, which is different only by a single spin flip at site i, is given by $1-q$ if the flipping follows the majority rule among the nearest neighbors of i, and q if it does not. When there is a tie in the neighbors' spin values, the transition rate is $1/2$. These transition rates can be written as

$$w_{MV}(\{\sigma\} \to \{\sigma'\}) = \frac{1}{2}\left[1 - \sigma_i S\left(\sum_{\langle j\rangle} \sigma_j\right)(1-2q)\right], \quad (1)$$

where $S(x)$ is the signum function for $x \neq 0$ and is zero if $x = 0$, and $\langle j\rangle$ denotes the nearest neighbors of i.

B. Transition rates

We can define the local "configuration energy" in the majority voter model $\varepsilon_{\mathrm{MV}}[\sigma_i]$ as

$$\varepsilon_{\mathrm{MV}}[\sigma_i] = -\sigma_i S\left(\sum_{\langle j\rangle} \sigma_j\right). \quad (2)$$

The minus sign in Eq. (2) makes the spin value of site i follow the majority of nearest-neighbor spins. With the configuration energy defined, we can consider different transition rates based on it, which specify how a configuration $\{\sigma\}$

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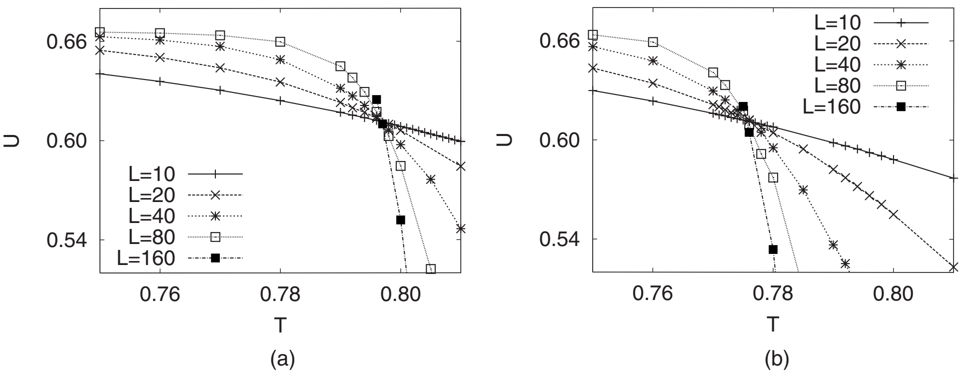

FIG. 1. Binder's fourth-order cumulant U as a function of temperature for the majority voter model with the Glauber rate (a) and the Metropolis rate (b).

evolves into a new one $\{\sigma'\}$. Here, we consider two of them, which we will call the Glauber rates $w_{\rm GL}(\{\sigma\}\to\{\sigma'\})$ and the Metropolis rates $w_{\rm ME}(\{\sigma\}\to\{\sigma'\})$, borrowing from the language of equilibrium statistical mechanics.

Glauber rate. In Glauber dynamics [22], the transition rate from a configuration $\{\sigma\}$ to another $\{\sigma'\}$ is given by $w_{\rm GL}=1/(1+e^{\beta_T\Delta})=[1-\tanh(\beta_T\Delta/2)]/2$, where β_T is the inverse temperature $1/k_BT$ and Δ is the energy difference between configurations after and before the spin flip. Since the energy difference takes values $+2,0,-2$ with the energy defined in Eq. (2) in the majority voter model, the Glauber transition rates of the majority voter model can be written as

$$w_{\rm GL}=\frac{1}{2}\left[1-\sigma_i S\left(\sum_{\langle j\rangle}\sigma_j\right)\tanh\beta_T\right]. \tag{3}$$

Comparing this expression with Eq. (1), we immediately see the correspondence between the original majority voter model and that with Glauber dynamics with configuration energy Eq. (2), which leads to the relation between the noise parameter q and the temperature in Glauber dynamics as $(1-2q)=\tanh\beta_T$.

Metropolis rate. For this choice of transition rate, spin flipping is controlled by the Metropolis rate [23]

$$w_{\rm ME}=\min[1,e^{-\beta_T\Delta}]. \tag{4}$$

Even though the transition rates differ from the original majority voter model, the Metropolis rates still capture the essential feature of the model, which favors the majority opinion, with a noise factor allowing minority opinion. In equilibrium systems, the choice of transition rate does not affect the critical behavior as long as the rates satisfy the detailed balance. In nonequilibrium systems, however, there is no such general rule, and it is an interesting question whether a change in transition rate affects the critical behavior or not.

C. Finite size scaling

The order parameter m and the susceptibility χ are defined as follows:

$$m=\frac{1}{N}\left|\sum_{i=1}^{N}\sigma_i\right|, \tag{5}$$

$$\chi=\frac{N}{T}(\langle m^2\rangle-\langle m\rangle^2). \tag{6}$$

Binder's fourth-order cumulant [21] is given by

$$U=1-\frac{\langle m^4\rangle}{3\langle m^2\rangle^2}. \tag{7}$$

For a square lattice with linear dimension L, the finite size scaling form of the order parameter, the susceptibility, and the derivative of Binder's fourth-order cumulant with respect to temperature are given by

$$m(L)=L^{-\beta/\nu}\tilde{m}(L^{1/\nu}t)\quad (t<0), \tag{8}$$

$$\chi(L)=L^{\gamma/\nu}\tilde{\chi}(L^{1/\nu}t), \tag{9}$$

$$U'(L)=L^{1/\nu}\tilde{U}'(L^{1/\nu}t), \tag{10}$$

where we write the reduced temperature as $t=(T-T_c)/T_c$.

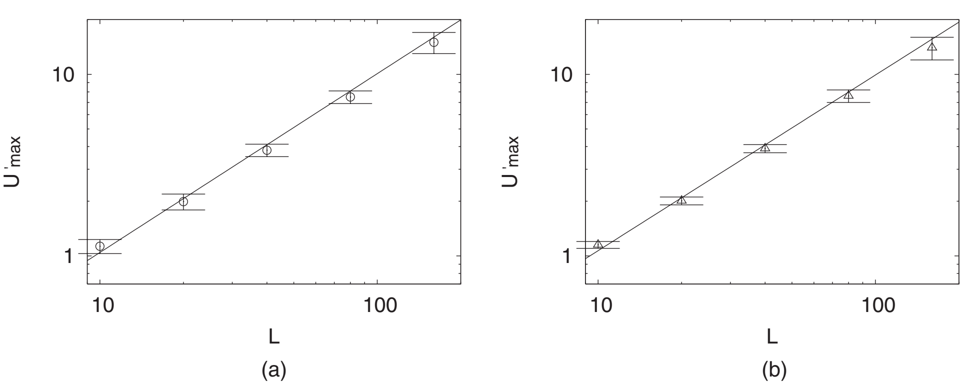

FIG. 2. Ln-ln plots of $U'(L)$ as a function of the linear dimension L for the majority voter model with the Glauber rate (a) and the Metropolis rate (b).

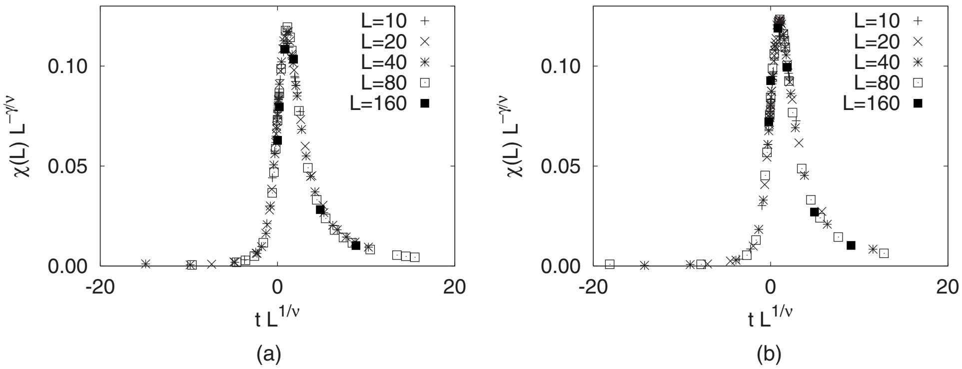

FIG. 3. $\chi(L)L^{-\gamma/\nu}$ as a function of $tL^{1/\nu}$ for the majority voter model with the Glauber rate (a) and the Metropolis rate (b).

According to finite size scaling theory [24], the scaling functions $\tilde{m}$, $\tilde{\chi}$, and $\tilde{U}'$ are universal and $m(L)$, $\chi(L)$, and $U'(L)$ are smooth and analytic functions in the vicinity of the critical temperature T_c.

D. Energy histogram

The normalized energy histogram $h(\varepsilon,\beta_T)$ is directly accumulated in the simulation, with respect to the energy function defined in Eq. (2) [20]. If we measure two energy histograms at different inverse temperatures β_T^1 and β_T^2, we can construct a simple histogram ratio for two different energies ε_1 and ε_2 as follows:

$$\mathcal{R}_H(\varepsilon_1,\varepsilon_2)=\frac{h(\varepsilon_1,\beta_T^1)}{h(\varepsilon_2,\beta_T^1)}\times\frac{h(\varepsilon_2,\beta_T^2)}{h(\varepsilon_1,\beta_T^2)}. \tag{11}$$

For the equilibrium model, $h(\varepsilon,\beta_T)$ is related to the Boltzmann distribution as follows:

$$h(\varepsilon,\beta_T)=g(\varepsilon)e^{-\beta_T\varepsilon}/Z(\beta_T), \tag{12}$$

where $g(\varepsilon)$ is the density of states and $Z(\beta_T)$ is the canonical partition function. Then the histogram ratio becomes a simple exponential:

$$\mathcal{R}=\exp[-(\beta_T^1-\beta_T^2)(\varepsilon_1-\varepsilon_2)]. \tag{13}$$

Since Eq. (12) does not hold for the nonequilibrium model, the normalized configurational energy histogram for the majority voter model obtained directly from simulations will deviate from Eq. (13). We will compare the histogram ratio for the majority voter model with Eq. (13) in Sec. III.

III. RESULTS

Simulations are carried out on an $L\times L$ square lattice with periodic boundary conditions for the majority voter model with the Glauber and the Metropolis rates, respectively. The linear dimension L is set as 10, 20, 40, 80, and 160. The location of the critical temperature for the majority voter model with each transition rate is estimated from Binder's fourth-order cumulants Eq. (7).

Figure 1 shows scaling plots of Binder's fourth-order cumulant as a function of temperature for the majority voter model with the Glauber rate [Fig. 1(a)] and the Metropolis rate [Fig. 1(b)]. The critical temperatures exist to characterize the order-disorder phase transition in the model [12]. We find that the critical temperature T_c for each transition rate is different. The critical temperature for the Glauber rate is higher than for the Metropolis rate. We get $T_c=0.796\pm0.005$ for the Glauber rate and $T_c=0.776\pm0.005$ for the Metropolis rate. Using $q=(1-\tanh\beta_T)/2$ given below Eq. (1), one can easily show that $T_c=0.796\pm0.005$ for the Glauber rate is the same as $q_c=0.075\pm0.001$ [5].

The scaling form of $U'(L)$ is given in Eq. (10). If we draw the ln-ln plot of the maximum of $U'(L)$ versus L, the slope gives the value of $1/\nu$. Figure 2 shows a fitting line with the Glauber rate [Fig. 2(a)] and the Metropolis rate [Fig. 2(b)]. The estimated values are $\nu=1.02\pm0.03$ for the Glauber rate and $\nu=1.03\pm0.03$ for the Metropolis rate.

We also estimate γ/ν in Eq. (9) from the scaling plots as a function of $tL^{1/\nu}$ (Fig. 3). We get $\gamma/\nu=1.78\pm0.05$ and 1.76 ± 0.05 for the Glauber and the Metropolis rates, respectively.

The critical exponent β of the order parameter is defined below the critical temperature and the scaling relation is

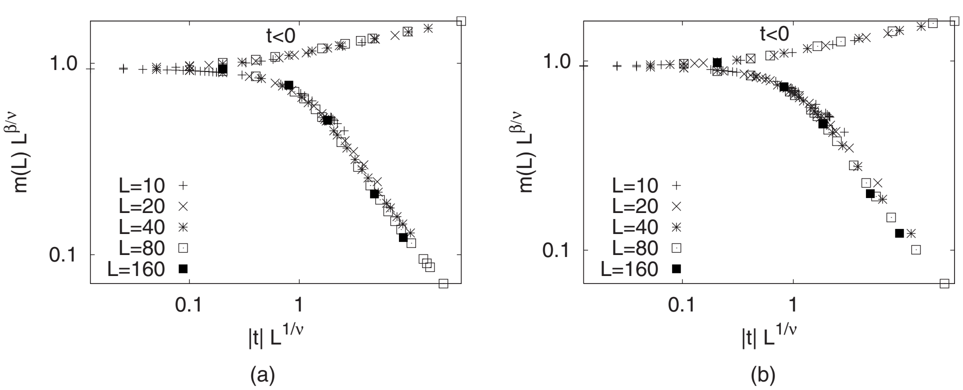

FIG. 4. $m(L)L^{\beta/\nu}$ as a function of $|t|L^{1/\nu}$ for the majority voter models with the Glauber rate (a) and the Metropolis rate (b).

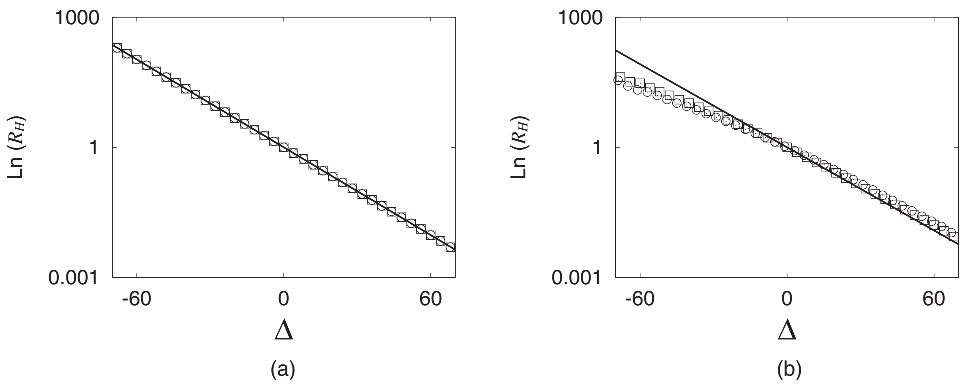

FIG. 5. Accumulated histogram ratio [Eq. (11)] and the theoretical histogram ratio [Eq. (13), solid line] for the Ising model (a) and the majority voter model (b) with Glauber (square) and Metropolis (circle) rates on a 20×20 system. We use $T_1=2.30$ and $T_2=2.80$ for the Ising model, and $T_1=0.80$ and $T_2=0.85$ for the majority voter model.

given by Eq. (8). From Fig. 4, which is the ln-ln plot of the scaling function $m(L)L^{\beta/\nu}$ as a function of scaling variable $L^{1/\nu}|t|$, we obtain $\beta/\nu=0.120\pm0.005$ for the Glauber rate and $\beta/\nu=0.123\pm0.005$ for the Metropolis rate. Our simulation results for the majority voter model indicate that the set of critical exponents are the same regardless of the type of transition rate, within numerical uncertainty.

Finally, we compare the energy histogram ratio directly obtained in simulations with that of the theoretical ratio for the equilibrium case, Eq. (13). For the Ising model, we choose two temperatures above T_c, in which one is slightly higher than the other. Figure 5(a) for the Ising model shows that the histogram ratios obtained directly from simulations coincide with the theoretical ratio $\mathcal{R}$ for both the Glauber and Metropolis rates, as expected. However, for the majority voter model [Fig. 5(b)], the histogram ratios are obviously different from the theoretical ratio $\mathcal{R}$ for both the transition rates, indicating that the dynamics of the majority voter model does not follow the Boltzmann distribution regardless of the type of transition rate.

IV. CONCLUSIONS

Using Monte Carlo simulation, we have studied the majority voter model on a square lattice with two transition rates, the Glauber and the Metropolis rates. The set of critical exponents are found to be identical in the two cases. Thus, the critical behaviors and universality class do not depend on the choice of transition rate, but the critical temperature does depend on it.

The difference of critical temperatures between the Glauber rate and the Metropolis rate can be estimated as follows. The possible energy differences Δ for a spin flip are -2, 0, or $+2$ in the majority voter model. Just below the critical temperature, the possible energy change is highly restricted, to only one energy change ($\Delta=+2$), because the possibility of energy change with $\Delta\leqslant0$ diminishes rapidly. Therefore, we may approximate $\omega_{\mathrm{ME}}(\beta_{T_c}^{\mathrm{ME}})\approx\omega_{\mathrm{GL}}(\beta_{T_c}^{\mathrm{GL}})$ if $\Delta=+2$ to get the following relation between the two critical temperatures:

$$\beta_{T_c}^{\mathrm{ME}}\approx\frac{\ln(1+e^{+2\beta_{T_c}^{\mathrm{GL}}})}{+2}.$$

If we put $T_c^{\mathrm{GL}}=0.796\pm0.005$ obtained from our simulations, we obtain $T_c^{\mathrm{ME}}=0.772\pm0.004$ from the above relation. This value is in good agreement with our simulation result $T_c^{\mathrm{ME}}=0.776\pm0.005$.

The critical exponents obtained are $\nu=1.02\pm0.03$, $\gamma/\nu=1.78\pm0.05$, $\beta/\nu=0.120\pm0.005$ for the Glauber rate, and $\nu=1.03\pm0.03$, $\gamma/\nu=1.76\pm0.05$, $\beta/\nu=0.123\pm0.005$ for the Metropolis rate. These values are in good agreement with the values in the two-dimensional Ising universality class, $\nu=1$, $\gamma=7/4$, and $\beta=1/8$.

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